The Dialectica category & related structures

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(Hi)story

- Dialectica category (Valeria de Paiva, PhD 1988)

  Motivation: Gödel's Dialectica interpretation, 1958
  Girard's Linear Logic ~1986
  decompose logical connectives: => to ⊸ (linear implication), ! (of course)
  → categorical models for Linear Logic ↗ classical ↘ intuitionistic

- Lenses (asymmetric/monomorphic ~ 2003 Pierce/Schmitt
  2007 + Foster, Greenwald, Moore)

  Motivation: modeling bx transformations
  view-update problem (database theory since late '70s)

- Wiring diagrams (Spivak ~2013
  + Rupel, Vagner, Lerman, Schultz, CV, ...)

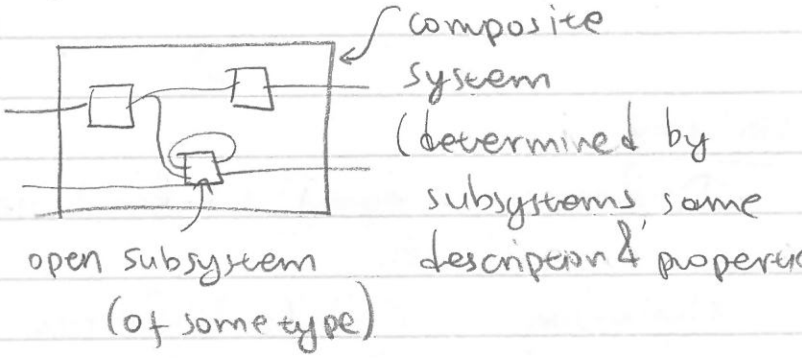
  Motivation: systems as operad algebras
  compositional analysis (zoom in & out, re-design)
  Moore machines, continuous dynamical systems,
  abstract machines ...

(*) Related: • open game theory (Hedges, Ghani, ... ~2015)
• learners (Fong, Spivak, Tuyeras ~ 2017)

→ 2017 Hyland: WD + Dialectica
2018 ACT Leiden: Lenses + WD (CV + joint project)

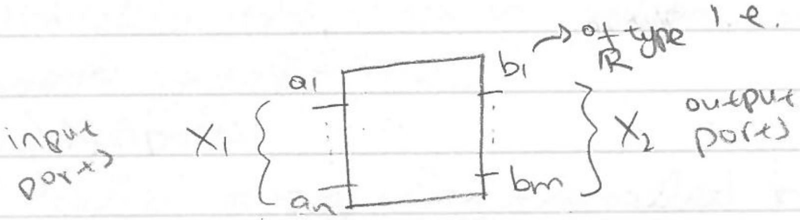
Wiring diagrams : categorical formalism for pictures like

→ Every possible interconnection of "boxes" is a morphism in a symmetric monoidal category (orthogonal to usual string theory pics! ⊞)



• For any $\mathcal{C}$, the category $W_{\mathcal{C}}$ of labelled boxes & wiring diagrams has

- objects pairs (X_1, X_2) of $\mathcal{C}$-typed finite sets, i.e. equipped with $\tau_i : X_i \to \mathrm{ob}\,\mathcal{C}$

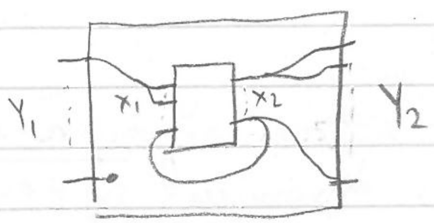


e.g. Set-labelled box

$\tau(a_1) = \mathbb{Z}$

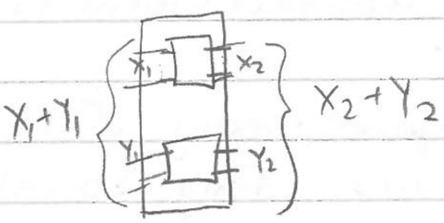
$\tau(a_n) = \{\perp, \top\}$ ...

- morphisms $(X_1, X_2) \to (Y_1, Y_2)$ pairs of functions $\begin{cases} X_1 \to X_2 + Y_1 \\ Y_2 \to X_2 \end{cases}$ that respect the types



"where the info comes from" (e.g. no passing wires)

- monoidal structure



$\emptyset \square \emptyset$

When $\mathcal{C}$ has products, this "maps" to a category $\hat{\mathcal{W}}_{\mathcal{C}}$
[idea: associate objects of $\mathcal{C}$, rather than sets, to input & output side of boxes!]

- objects are pairs $\Big(S = \underbrace{\prod_{x \in X_2} \tau_2(x)}, \; T = \prod_{x\in X_1} \tau_1(x)\Big) \in \mathcal{C}\times\mathcal{C}$

  ↳ product of all output types, e.g. $\mathbb{Z} \times \{\bot, \top\} \times \cdots \times \mathbb{N} \in \underset{\sim}{\mathcal{S}et}$

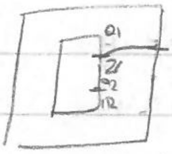
- morphisms $(S,T) \to (A,B)$ are

  ["taking products" is contravariant & strong monoidal]

  sum turned into product

  $\begin{cases} p : \ulcorner S \times B \urcorner \longrightarrow T \\ g : S \longrightarrow A \end{cases}$

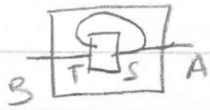
  $\prod_{x\in X_2} \tau(x) = S$, $A = \prod_{Y_2} \tau(y)$



$Y_2 \longrightarrow X_2$

$b_1 \longmapsto a_1$

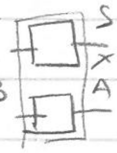
turns into $\underset{\mathbb{Z}}{\tau(b_1)} \xleftarrow{\pi_2} \underset{\mathbb{Z} \times \mathbb{R}}{\tau(a_1)\times\tau(a_2)}$



$$\begin{tikzcd} S \arrow[d, "g"'] & T \\ A & B \arrow[u, "p"'] \end{tikzcd}$$

Composition $\begin{tikzcd} S \arrow[d, "g"'] & T \\ A \arrow[d, "f"'] & B \arrow[u, "p"'] \\ P & R \arrow[u, "q"'] \end{tikzcd}$ is $\begin{cases} S \xrightarrow{g} A \xrightarrow{f} P \\ S \times R \xrightarrow{\Delta\times 1} S\times S\times R \xrightarrow{1\times g\times 1} S\times A\times R \xrightarrow{1\times q} S \times B \xrightarrow{P} T \end{cases}$

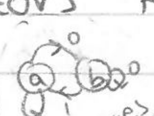
- monoidal structure



$\rightsquigarrow (\hat{\mathcal{W}}_{\mathcal{C}}, \otimes, \text{-}\square\text{-})$ is a symmetric monoidal category

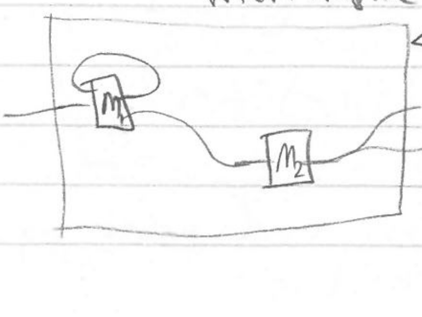
- Various systems are algebras on $\hat{\mathcal{W}}_{\mathcal{C}}$ (or the induced operad), i.e. lax monoidal (pseudo) functors $K : (\hat{\mathcal{W}}_{\mathcal{C}}, \otimes, \text{-}\square\text{-}) \longrightarrow (\underset{\sim}{\mathcal{C}at}, \times, 1)$

e.g. $K(S,T)$ = category of all Moore machines with input T & output S



$$\begin{tikzcd} (S,T) \arrow[r, mapsto] \arrow[d, "\text{wiring diagram}"'] & K(S,T) \arrow[d] & m \arrow[d] \\ (A,B) \arrow[r, mapsto] & K(A,B) & \underline{\underline{m'}} \end{tikzcd}$$

new machine



functoriality & monoidality of K produces a NEW M as their composite!

... completely expressed in terms of subsystems

Dialectica categories

Take $\mathcal{C}$ symmetric monoidal closed with products, $(\mathcal{C}, \otimes, I, \multimap, \times)$ $\multimap \to [-,-]$

→ $\mathcal{C}\times\mathcal{C}^{op}$ has monoidal structure

$$(S,T)\otimes(A,B) := \big(S\otimes A,\ (A\multimap T)\times(S\multimap B)\big)$$

in fact, it's closed:

$$\frac{\big(S\otimes A,\ (A\multimap T)\times(S\multimap B)\big) \longrightarrow (P,R)}{(S,T)\longrightarrow\big((A\multimap P)\times(R\multimap B),\ R\otimes A\big)}$$

■ $G(\mathcal{C}) = \mathcal{C}\times\mathcal{C}^{op}$ is a categorical model of Classic Linear Logic

In particular, it has products $(S\times A, T+B)$ | when $\mathcal{C}$ has
& coproducts $(S+A, T\times B)$ | moreover coproducts

Suppose $\mathcal{C}$ is CCC

→ There is a comonad $F : \mathcal{C}\times\mathcal{C}^{op} \longrightarrow \mathcal{C}\times\mathcal{C}^{op}$

$$(S,T) \longmapsto (S, T^S)$$

comultiplication: $F(S,T) \longrightarrow FF(S,T)$ in $\mathcal{C}\times\mathcal{C}^{op}$

(S, T^S) $\qquad (S, (T^S)^S) \cong (S, T^{S\times S})$

is $\begin{cases} S \xrightarrow{id} S \\ \multimap\Delta : T^{S\times S} \longrightarrow T^S \end{cases}$

counit: $F(S,T) \to (S,T)$ $\qquad$ is $\begin{cases} S \xrightarrow{id} S \\ \multimap ! : T \longrightarrow T^S \end{cases}$

(S, T^S) $\qquad\qquad \dfrac{T\longrightarrow T^S}{S\times T \xrightarrow{\pi_2} T}$

▲ The Dialectica category $D(\mathcal{C})$ is the coKleisli category $(\mathcal{C}\times\mathcal{C}^{op})_F$,

i.e. has the same objects (S,T), & morphisms $(S,T) \rightsquigarrow (A,B)$ are

$F(S,T) = (S, T^S) \longrightarrow (A,B)$ in $\mathcal{C}\times\mathcal{C}^{op}$, namely

$$\begin{cases} S \longrightarrow A \\ \dfrac{B \longrightarrow T^S}{S\times B \longrightarrow T} \end{cases}$$

■ $\boxed{D(\mathcal{C}) \cong \hat{\mathcal{W}}_{\mathcal{C}}}$

■ $D(\mathcal{C})$ is (...) a categorical model for (the propositional part of) Intuitionistic Linear Logic

weak coproducts!

Remarks

- $D(-)$ & $\hat{\mathcal{W}}_{(-)}$ are functorial

$$(\text{ccCat})\ \underset{\sim}{\text{fcCat}} \longrightarrow \underset{\sim}{\text{SymMonCat}}$$

$$\begin{tikzcd} \mathcal{C} \arrow[d, "F"'] \arrow[r, mapsto] & \hat{\mathcal{W}}_{\mathcal{C}} \arrow[d, shift right] & (S,T) \arrow[d] \\ \mathcal{D} & \hat{\mathcal{W}}_{\mathcal{D}} \arrow[u, shift right] & (FS,FT) \end{tikzcd}$$

with $(S\times A, T\times B)$ which is not the categorical product!

- In original work, $D(\mathcal{C})$ has objects "relations" $U \xrightarrow{\alpha} S\times T$, and morphisms

$$\begin{tikzcd} S \arrow[r, "a", "+"'] \arrow[d, "g"'] & T \\ A \arrow[r, "+", "b"'] & B \arrow[ul, "p"'] \end{tikzcd}$$

such that a non-trivial condition is satisfied

$$\begin{tikzcd} & \bullet \arrow[r] \arrow[d, tail] \arrow[dl, dashed, "\exists! k"'] & U \arrow[d, tail, "\alpha"] \\ \bullet \arrow[r, tail] \arrow[d] & S\times B \arrow[r, "(\pi_1, p)"'] \arrow[d, "g\times 1"] & S\times T \\ V \arrow[r, tail, "\beta"'] & A\times B & \end{tikzcd}$$

"whenever $s\ \alpha\ p(s,b)$, (relation)
then $g(s)\ \beta\ b$"

→ In easier $G(\mathcal{C})$, with $\begin{tikzcd} S \arrow[r, "a", "+"'] \arrow[d, "g"'] & T \\ A \arrow[r, "+", "b"'] & B \arrow[u, "f"'] \end{tikzcd}$,
$s\ \alpha\ f(b) \leq_{\Omega} g(s)\ \beta\ b$ (lineale) is a "semi-adjointness" condition

- Standard coKleisli theory $\rightsquigarrow$

$$\begin{tikzcd} \mathcal{C}\times\mathcal{C}^{op} \arrow[r, bend left, "\text{"}F\text{"}"] \arrow[r, phantom, "\perp"] & D(\mathcal{C}) = (\mathcal{C}\times\mathcal{C}^{op})_F \arrow[l, bend left] \end{tikzcd}$$

$$\begin{tikzcd} (S,T) \arrow[r, mapsto] \arrow[d, "f"'] & (S,T) \arrow[d, "f"'] \\ (A,B) \arrow[u, shift right, "g"'] & (A,B) \arrow[u, "\text{via } S\times B \xrightarrow{\pi_B} B \xrightarrow{g} T"'] \end{tikzcd}$$

$\underbrace{S\times B \xrightarrow{\pi_B} B}_{\text{counit}}$

E.g. $D(\mathcal{C})$ inherits products via right adjoint, $(S\times A, T+B)$, but no more limits or colimits are expected in general! [weak coproducts]

Lenses: interactions between a database & a view of it

Monomorphic / asymmetric lenses are two objects S, A in $\mathcal{C}$ (+products)

SOURCE (e.g. database) VIEW (e.g. results of query)

with

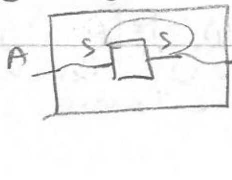
$$\begin{cases} p: S\times A \to S & \text{"put" – UPDATE} \\ g: S \longrightarrow A & \text{"get" – VIEW} \end{cases}$$

initial state of whole; updated state of zoomed-in; NEW state of whole

Clearly, this is a wiring diagram / dialectica morphism $(S,S) \to (A,A)$

$$\begin{tikzcd} S \arrow[d] & S \\ A \arrow[ur] & A \arrow[u] \end{tikzcd}$$

or



Well-behaved lenses satisfy

Put Get : $S \times A \xrightarrow{p} S \xrightarrow{g} A = \pi_A$

"YOU GET BACK WHAT YOU PUT IN"

Get Put : $S \xrightarrow{\Delta} S \times S \xrightarrow{1 \times g} S \times A \xrightarrow{p} S = id_S$

"PUTTING BACK WHAT YOU GOT DOESN'T CHANGE ANYTHING"

E.g. constant-complement view-updating lens

$(\underbrace{A_1 \times A_2}_{\text{source}}, \underbrace{A_1}_{\text{view}})$ by $\begin{cases} p : (A_1 \times A_2) \times A_1 \xrightarrow{\pi_{23}} A_1 \times A_2 & (a_1, a_2), a_1' \mapsto (a_1', a_2) \\ g : A_1 \times A_2 \xrightarrow{\pi_1} A_1 \end{cases}$

is a well-behaved lens [complement remains unchanged]

Bimorphic lenses $(S,T) \to (A,B)$ are precisely dialectica morphisms

$\begin{cases} S \times B \to T \\ S \to A \end{cases}$

↳ view can change to different type B, resulting to a change of the whole from S to type T.

$$\boxed{D(\mathcal{C}) \cong \hat{W}_{\mathcal{C}} \cong \text{Bilens}(\mathcal{C})}$$

OPEN QUESTIONS/DIRECTIONS : channel of communication between completely different areas serving different purposes.

- Transfer of structures, properties & intuition.
- $(S,S) \to (A,A)$ monomorphic lenses
  $(S,S) \to (A,B)$ Moore machines
  $(S,T) \to (A,B)$ Dialectica translation of implication
  } CLASSIFY/ UNDERSTAND SUBCATEGORIES OF INTEREST
- Conditions for relations in Dialectica & functoriality of abstract systems/contracts algebra suspiciously similar. Also, some algebras themselves expressed as Dialectica maps....
- $D(\mathcal{C})$ monoidal closed via $[(S,T),(A,B)] = (A^S \times T^{S \times A}, S \times B)$. What does it mean for W, D & Bilens?
- Spans incorporated
  ↗ symmetric lenses ≡ spans of asymmetric
  ↘ abstract machines are span-like algebras